A SUSPENSION FLOW OF BLOOD THROUGH A BELL SHAPED STENOSIS

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Abstract- The present work deals with the flow of blood through a bell shaped stenosis assuming that the flowing flood is represented by a macroscopic two-phase model (i.e., a suspension of erythrocytes in plasma). The coupled differential equations describing the flow of fluid (plasma) and the particle (red cell) phases have been solved and the expressions for the flow characteristics, namely, the impedance, the wall shear stress in the stenotic region, the shear stress at the stenosis throat have been derived. Results obtained in the paper have been discussed graphically in brief.

Key Words: Hematorit, impedance, shear stress, stenosis throat, stenosis .

INTRODUCTION

The narrowing of any body passage, tube or orifice in a living mammal is known as stenosis or arteriosclerosis, an abnormal and unnatural growth that develops at various locations of the cardiovascular systems under diseased conditions and occasionally results in to serious consequences (cerebral strokes, myocardial infarction, angina pectoris, cardiac arrests, etc.). Probably the deposits of the cholesterol, fatty substances, cellular waste products, calcium and fibrin in the inner lining of the of an artery, etc. are responsible for the frequently occurring disease. Irrespective of the cause, it is well known that once the constriction has developed, it brings about the significant changes in the flow field. The knowledge that the hemodynamic factors play an important role in the genesis and proliferation of the disease has attracted the investigators including Mann et al. (1938), Young (1968, 1979), Young and Tsai (1973), Caro et al. (1978), Shukla et al. (1980), Ahmed and Giddens (1983), Sarkar and Jayaraman (1998), Pralhad and Schultz (2004), Jung et al. (2004), Liu et al. (2004), Srivastava and coworkers (1996, 2009, 2010), Mishra et al. (2006), Misra and Verma (2007), Ponalagusamy (2007), Layek et al. (2005, 2009), Joshi et al. (2009), Mekheimer and El-Kot (2008), Tzirtzilakis (2008), Mandal and coworkers (2005, 2007a,b), Politis et al. (2007, 2008), Singh et al. (2010), Srivastava et al. (2011), Mishra et al. (2011) and many others.

At low shear rates being a suspension of corpuscles, blood behaves like a non-Newtonian fluid. However, the theoretical study of Haynes (1960) and experimental observations of Cokelet (1972) indicate that blood can not be treated as a single-phase homogeneous viscous fluid in narrow arteries (of diameter $\leq 1000\,\mu$m). Srivastava and Srivastava (1983) presented a brief review of the modeling of blood flow and observed that the individuality of red cells (of diameter 8μm) is important even in such large vessels with diameter up to 100 cells diameter. A brief survey of the literature on multiphase blood flow has recently been presented by Srivastava (2007). A survey of the literature on arteriosclerotic development further reveals that the studies conducted are mainly concerned with the single symmetric and non-symmetric stenoses. An attempt is made in the present investigation to explore the effects of a bell shaped stenosis on the flow characteristics of blood taking into account that the flowing blood is to be treated as macroscopic two-phase fluid (i.e., a suspension of erythrocytes in plasma).

FORMULATION OF THE PROBLEM

Consider the axisymmetric flow of blood through a bell shaped stenosis, specified at the position shown in Fig.1, in an artery with permeable wall. The geometry of the stenosis which is assumed to be manifested in the arterial wall segment is described as

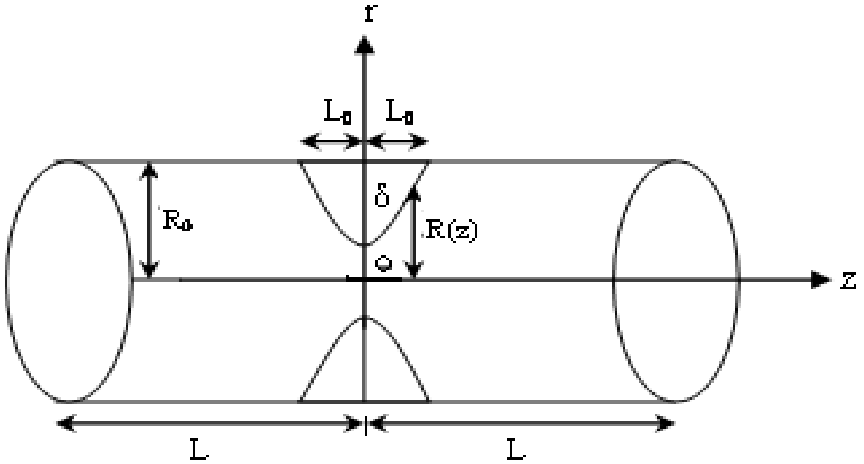

Fig.1 The geometry of a bell shaped stenosis with permeable wall.

$$\frac{R(z)}{R_0} = 1 - \frac{\delta}{R_0}\exp\left(\frac{-m^2\varepsilon^2 z^2}{R_0^2}\right), \tag{1}$$

where R_0 is the radius of the arterial segment in the non-stenotic region, R(z) is the radius of the stenosed portion of the arterial segment located at the axial distance z from the left end of the segment, δ is the depth of stenosis at the throat into the lumen and m is a parametric constant, ε the relative length of the constriction, defined as the ratio of the radius to the half length of the stenosis, i.e., $\varepsilon = R_0/L_0$.

Blood is assumed to be represented by a macroscopic two-phase model, that is, a suspension of erythrocytes (red cells) in plasma. The equations describing the steady flow of a two-phase macroscopic model of blood may be expressed (Srivastava and Srivastava, 1983, 1989) as

$$(1\text{-}C)\,\rho_f \left\{ u_f \frac{\partial u_f}{\partial z} + v_f \frac{\partial u_f}{\partial r} \right\} = -(1\text{-}C)\frac{\partial p}{\partial z} + (1-C)\,\mu_s(C)\,\nabla^2 u_f + CS(u_p - u_f), \tag{2}$$

$$(1\text{-}C)\,\rho_f \left\{ u_f \frac{\partial v_f}{\partial z} + v_f \frac{\partial v_f}{\partial r} \right\} = (1\text{-}C)\frac{\partial p}{\partial r} + (1-C)\,\mu_s(C)\,\left(\nabla^2 - \frac{1}{r^2}\right) v_f + CS(v_p - v_f), \tag{3}$$

$$\frac{\partial}{\partial r}\left[(1-C)v_f\right] + (1-C)\frac{v_f}{r} + \frac{\partial}{\partial z}\left[(1-C)u_f\right] = 0, \tag{4}$$

$$\rho_p \left\{ u_p \frac{\partial u_p}{\partial z} + v_p \frac{\partial u_p}{\partial r} \right\} = -C \frac{\partial p}{\partial z} + CS\,(u_f - u_p), \qquad (5)$$

$$\rho_p \left\{ u_p \frac{\partial v_p}{\partial z} + v_p \frac{\partial v_p}{\partial r} \right\} = -C \frac{\partial p}{\partial r} + CS\,(v_f - v_p), \qquad (6)$$

$$\frac{\partial}{\partial r}[C\,v_p] + \frac{C v_p}{r}\,\frac{\partial [C u_p]}{\partial z} = 0, \qquad (7)$$

with $\nabla^2 = \partial/\partial r^2 + (1/r)(\partial/\partial r) + \partial^2/\partial z^2$ as Laplacian operator, r is the radial coordinate measured perpendicular to the axis of the tube. (u_f, v_f) and (u_p, v_p) are the (axial, radial) components of the fluid particle velocities, respectively, ρ_f and ρ_p are the actual density of the material constituting the fluid (plasma) and the particle (erythrocyte) phases, respectively, (1-C) ρ_f is the fluid phase and $C\rho_p$ is particle phase densities, C denotes the volume fraction density of the particles, p is the pressure, $\mu_s(C) \simeq \mu_s$ is the mixture viscosity (apparent or effective viscosity) , S is the drag coefficient of interaction for the force exerted by one phase on the other, and the subscripts f and p denote the quantities associated with the plasma (fluid) and erythrocyte (particle) phases, respectively. Others limitations of the present model are the same as discussed in Srivastava and Srivastava (2009). The expressions for drag coefficient of interaction, S and the viscosity of the suspension, μ_s for the present study are selected (Srivastava and Srivastava, 2009) as

$$S = \frac{9}{2}\frac{\mu_o}{a_o^2}\;\frac{4 + 3[8C - 3C^2]^{1/2} + 3C}{(2 - 3C)^2}\;, \qquad (8)$$

$$\mu_s(C) = \frac{\mu_o}{1 - mC}\;,$$

$$m = 0.070 \exp\,[2.49C + (1107/T) \exp(-1.69C)], \qquad (9)$$

where T is the measure in absolute scale of temperature (K) , μ_0 is the constant plasma viscosity and a_0 is the radius of an erythrocyte.

It seems to be a formidable task to obtain the solution of equations (2)-(7). Depending however, on the size of the stenosis, certain terms in theses equations are of less significance than others. Now following the reports of Young (1968), Srivastava Srivastava (2009), the equations governing the laminar, steady, one-dimensional flow of blood in an artery in the case of a mild stenosis (i.e., $\delta/R_0 \ll 1$) are derived from equations (2)-(7) as

$$(1\text{-}C)\frac{dp}{dz} = (1 - C)\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) u_f + CS\,(u_p - u_f), \qquad (10)$$

$$C\frac{dp}{dz} = CS\,(u_f - u_p)\,, \qquad (11)$$

The boundary conditions are

$$u_f = 0 \quad \text{at} \quad r = R(z) \tag{12}$$

$$\frac{\partial u_f}{\partial r} = 0 \quad \text{at} \quad r = 0. \tag{13}$$

ANALYSIS

An integration of equations (10) and (11), subject to the boundary conditions (12) and (13), yields the expressions for the velocity of fluid and particle phases as

$$u_f = \frac{-R_0^2\, dp/dz}{4(1-C)\mu_s}\left[(R/R_0)^2 - (r/R_0)^2\right], \tag{14}$$

$$u_p = \frac{-R_0^2\, dp/dz}{4(1-C)\mu_s}\left[(R/R_0)^2 - (r/R_0)^2 + \frac{4(1-C)\mu_S}{SR_0^2}\right]. \tag{15}$$

The flow flux, Q is now calculated as

$$\begin{aligned} Q &= 2\pi\left\{\int_0^R r\left[(1-C)u_f + C\,u_p\right]dr\right\} \\ &= -\frac{\pi R_0^4\, dp/dz}{8(1-C)\mu_S}\left[(R/R_0)^4 + \beta(R/R_0)^2\right], \end{aligned} \tag{16}$$

where $\beta = 8C(1-C)\mu_s/SR_0^2$, a non –dimensional suspension parameter.

The pressure drop, $\Delta p\ (= p \text{ at } z = \text{-}L, – p \text{ at } z = L)$ across the stenosis in a tube of length 2L is calculated from equation (16) as

$$\Delta p = \int_{-L}^{L}\left(-\frac{dp}{dz}\right)dz = \frac{8(1-C)\mu_s Q}{\pi R_0^4}\psi, \tag{17}$$

where

$$\psi = \int_{-L}^{-L_0}[\varphi(z)]_{R/R_0=1}dz + \int_{-L_0}^{L_0}\varphi(z)dz + \int_{L_0}^{L}[\varphi(z)]_{R/R_0=1}dz,$$

$$\varphi = 1/[(R/R_0)^4 + \beta(R/R_0)^2]. \tag{18}$$

The first and the third integrals in the expression for ψ obtained above are straight forward whereas the analytical evaluation of the second integral is almost a formidable task and thus will be evaluated numerically. Following now the reports of Young (1968) and Srivastava and Rastogi (2009), one obtains the expressions for impedance, λ, the wall shear stress in the stenotic region, τ_w, the shear stress at stenosis throats, τ_s in their non-dimensional form as

$$\lambda = (1-C)\mu\left[\frac{1-L_0/L}{1+\beta} + \frac{1}{L}\int_{-L_0}^{L_0}\frac{dz}{\left[(R/R_0)^4 + \beta(R/R_0)^2\right]}\right], \tag{19}$$

$$\tau_w = \frac{(1-C)\mu}{\left[R/R_0{}^3 + \beta(R/R_0)\right]}, \tag{20}$$

$$\tau_s = \frac{(1-C)\mu}{\left[(1-\delta/R_0)^3 + \beta(1-\delta/R_0)\right]}, \tag{21}$$

where $\lambda = \bar{\lambda}/\lambda_0,\ (\tau_w, \tau_s) = \dfrac{(\bar{\tau}_w, \bar{\tau}_s)}{\tau_0},$

$$\bar{\lambda} = \frac{\Delta p}{Q},\ \bar{\tau}_W = -\frac{R}{2}\left(\frac{dp}{dz}\right),\ \bar{\tau}_S = \left[-\frac{R}{2}\left(\frac{dp}{dz}\right)\right]_{R/R_0 = (1-\delta/R_0)},$$

$$\mu = \mu_s/\mu_0, \lambda_0 = 16\mu_0 L/\pi R_0^4,\ \tau_0 = 4\mu_0 Q/\pi R_0^3,$$

λ_0 and τ_0 are respectively the flow resistance and wall shear stress for a normal artery (no stenosis) in the absence of the particle phase (i.e. C=0, Newtonian fluid).

In the absence of the particles (i.e. C=0) the results for a Newtonian fluid are derived from equation (19)-(21), as

$$\lambda_N = 1 - \frac{L_0}{L} + \frac{1}{L}\int_d^{d+L_0}\frac{dz}{(R/R_0)^4}, \tag{22}$$

$$\tau_{wN} = \frac{1}{(R/R_0)^3}, \tag{23}$$

$$\tau_{sN} = \frac{1}{(1-\delta/R_0)^3}, \tag{24}$$

where the subscript N stands for the Newtonian fluid.

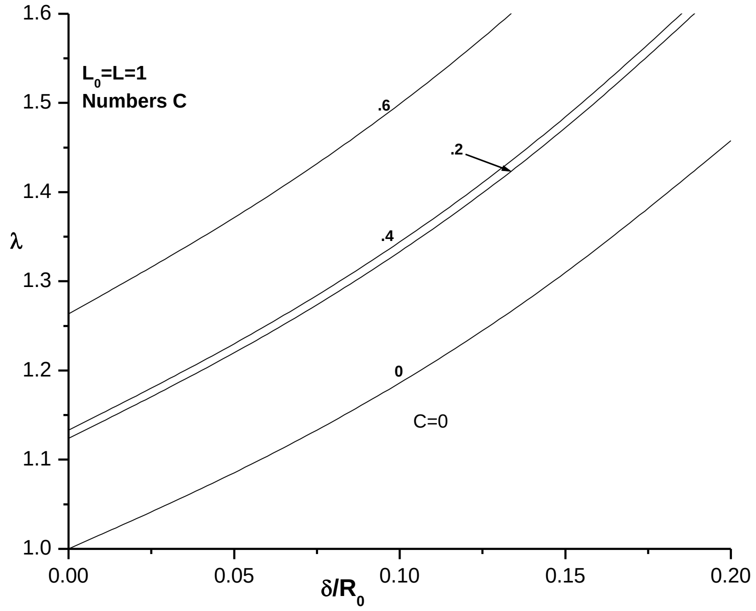

Fig. 2 λ **versus** δ/R_0 **for different C.**

NUMERICAL RESULTS AND DISCUSSIONS

In order to discuss the results of the study quantitatively, computer codes are developed to evaluate analytical results obtained in equations (19)-(21) at the temperature of 37^0 C in an artery of radius 0.01cm for various parameter values selected from Young (1968), Srivastava and Srivastava (2009) and Srivastava and Rastogi (2009). The parameter values are: L_0 (cm) = 1; L(cm) = 1, 2, 5 ; C = 0, 0.2, 0.4, 0.6; δ/R_0 =0, 0.05, 0.10, 0.15, 0.20.

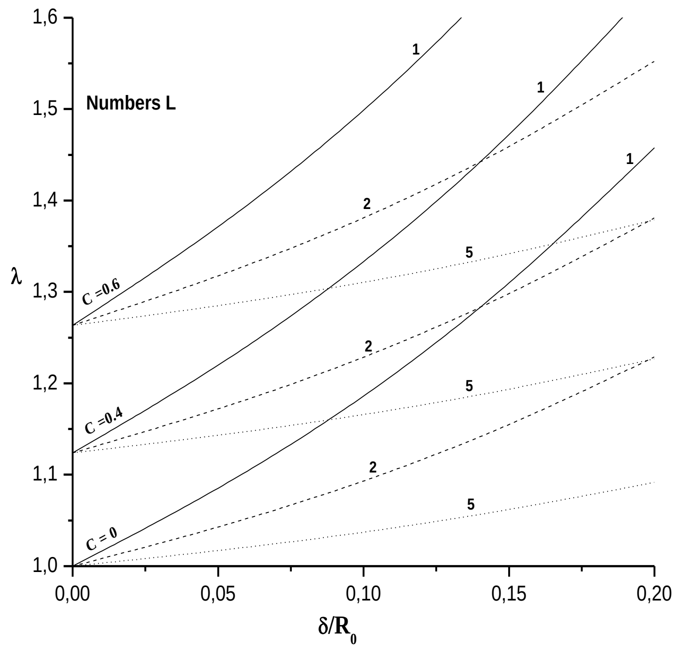

Fig. 3 λ **versus** δ/R_0 **for different L and C.**

The resistance to flow, λ increases with the hematocrit, C as well as with the stenosis height, δ/R_0 (Fig.2). The impedance, λ decreases with the increasing length of the tube which in terns implies that the impedance, λ increases with the stenosis length, L_0 (Fig.3). The blood flow characteristic, λ increases steeply with the hematorcit, C for any given set of other parameters (Fig.4).

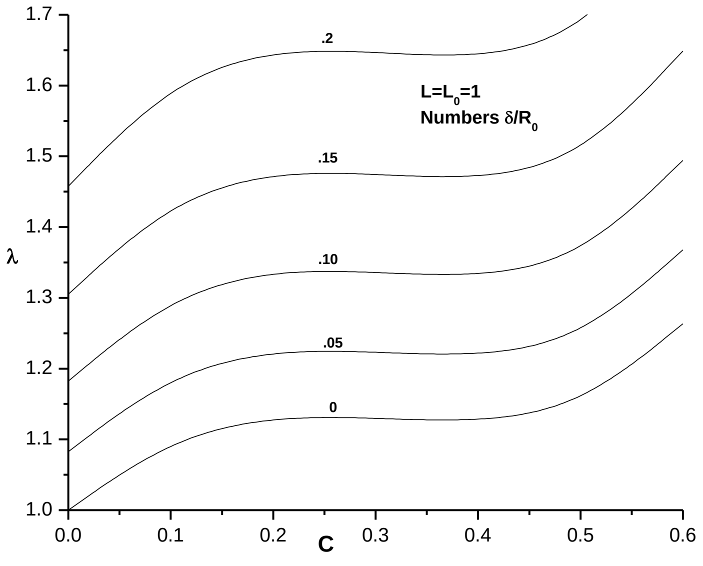

Fig. 4 λ versus C for different δ/R_0.

At any axial distance the wall shear stress in the stenotic region, τ_w increases with the hematocrit, C and stenosis height, δ/R_0 (Figs. 5& 6). The blood flow characterstic, τ_w increases

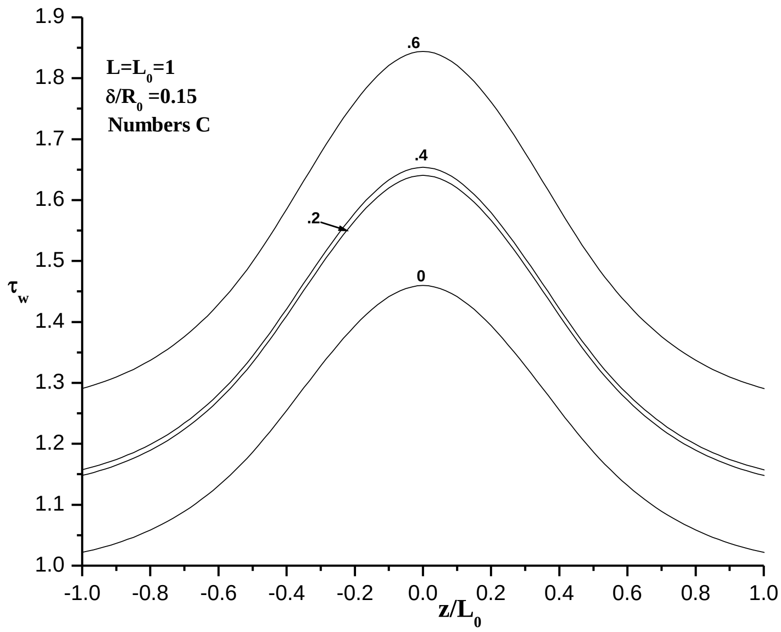

Fig. 5 τ_w versus z/L_0 for different C.

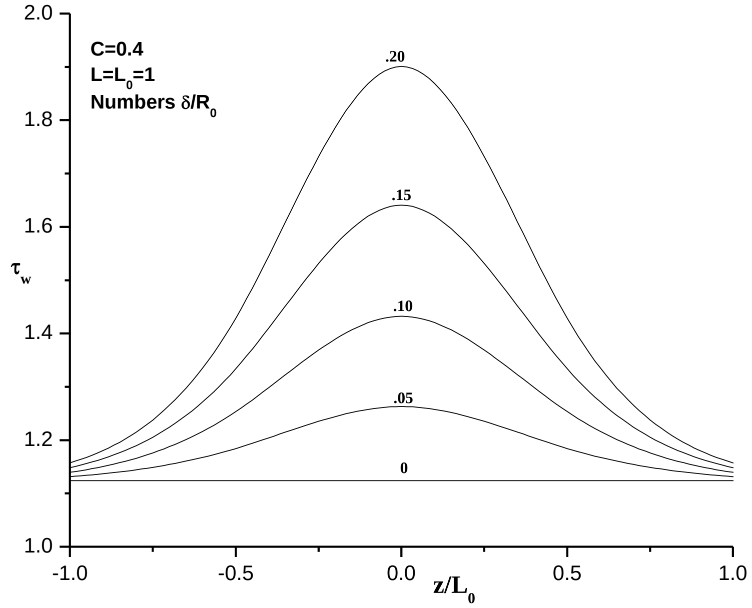

Fig. 6 τ_w **versus** z/L_0 **for different** δ/R_0.

rapidly in the up stream of the stenosis throat and attains its peak magnitude at the throat located at

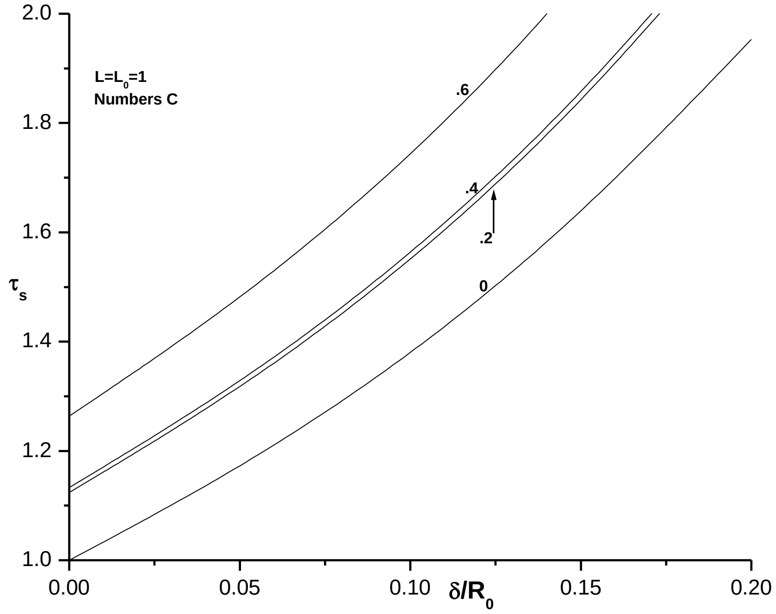

Fig. 7 τ_s **versus** δ/R_0 **for different C.**

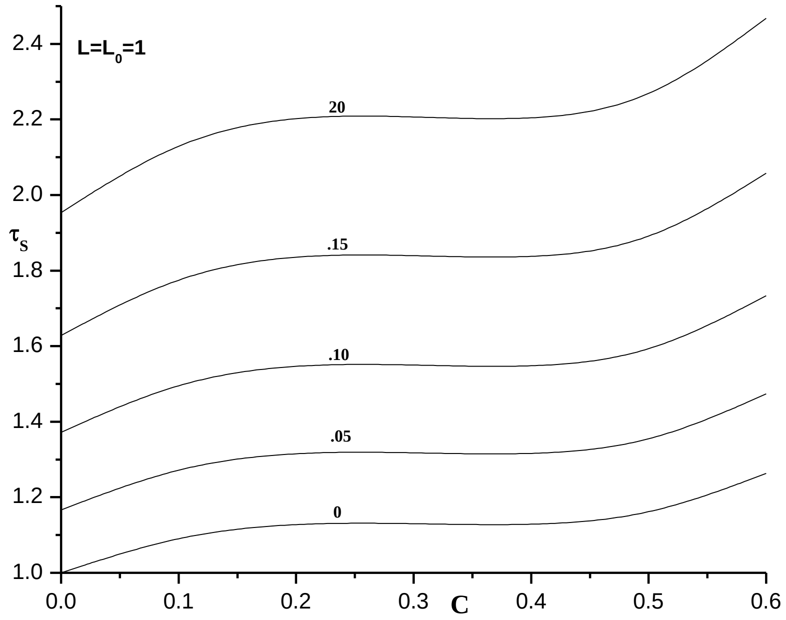

Fig. 8 τ_s versus C for different δ/R_0.

z/L_0 =0, it then decreases rapidly in the down stream of the throat and achieves its approached value (i.e., at $z/L_0 = -1$) at the end point of the constriction profile located at z/L_0 =1 (Fig. 6). The shear stress at the stenosis throat, τ_s also increases with the hematocrit, C and the stenosis height, δ/R_0 (Fig. 7). An inspection of Figs. 2-4, 7 and 8 reveals that the shear stress at the stenosis throat, τ_s possesses the characterstics similar to that of the flow resistance, λ with respect to any parameter.

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